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Engineering Notes

Gain Scheduled Control: Switched Polytopic System Approach

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I. Introduction

GAIN-SCHEDULED controllers are the most common and practical class of flight control systems for aerospace plants, due to its high reliability and many design experiences accumulated in the past decades [1–9]. Although the gain-scheduled control is quite important among the practitioners, it still exists that some poorly developed but important theoretical issues. One such issue is that relaxing the slow variation requirement for system parameters [2–5].

Conventional gain-scheduled control needs the parameters of the plant vary slowly for stability, but for most of modern aircrafts, especially for high performance fighters and hypersonic vehicles, the parameters vary fast so that gain-scheduled control method is no longer theoretically feasible. Up to now, the issue associated with relaxing the slow variation requirement includes two aspects [2]: one is relaxing the requirement of near equilibrium operation that results from the equilibrium linearizations of the plant, while the other is relaxing the requirement of slow variation between operating regions, which associates with ensuring the stability within the flight envelope.

For the former aspect, two new modeling approaches, fuzzy approach and neural approach, seem to be good ways to deal with the problem [2]. Besides, the notable work by Leith and his coworkers use a velocity-based analysis framework to relax the slow variation requirement [4]. This framework associates a linear system with every operating point of a nonlinear plant, not just the equilibrium points, and so is not restricted to near equilibrium operation.

With regard to the later, it is still an open question. For linear parameter-varying (LPV) gain-scheduled methods, arbitrarily fast variation for parameters is allowed by using a common Lyapunov function which guarantees the closed-loop system stability [5–7,10]. However, for most modern aircrafts, a common Lyapunov function probably does not exist due to a large parameter variation range within the full flight envelope [11]. Recent works by Wu and his coworkers propose a switched LPV/linear fractional transformation gain-scheduled method that uses multiple Lyapunov functions instead of a common Lyapunov function to guarantee the closed-loop system stability [10,12]. This method is less conservative and may achieve desired performance over a large parameter variation range. However, it leads to high computational complexity, and such a controller may not exist or may not be found even if one does exist [2,3,5,10]. Essentially, when performing the controller design, the aforementioned methods involve a linear matrix inequality (LMI)-based direct synthesis framework. This framework results in not only computational complexity and design conservativeness, but also lacking in usage of large amount of experiences on linear controller design accumulated during the past decades. Therefore, it is quite important to pursuit other technology to synthesize a gain-scheduled controller.

In this study, a gain-scheduled controller synthesis approach is proposed based on stability theory for switched systems evolving on locally overlapped switching law. Switched systems are consisted of a collection of subsystems, together with a switching law that specifies the switchings between the subsystems [13–15]. These systems have a wide range of applications in the aerospace field [16–22], particularly when the switching law is with the "locally overlapped" property [17,23–25]. For flight control, linear models on all operating points within the full envelope are subsystems of a switched system with locally overlapped switching law. The switching law is specified by continuous scheduling variables, such as aircraft altitude and Mach, and therefore can be partitioned into several locally overlapped parts. For example, the active subsystem on operating point A can only switch to an adjacent operating point B, before it switches to a further operating point C. In this situation, the switching law is partitioned into two locally overlapped groups, A-B and B-C. In the previous work of the authors, it is shown that a switched linear system with locally overlapped switching law is globally asymptotically stable, provided the average dwell time on common subsystems is no smaller than a fixed positive constant [17]. In this study, this result will be extended to switched polytopic system and thereby a gain-scheduled controller synthesis approach is proposed.

A switched polytopic system is established to describe the plant dynamics within the full flight envelope. Every polytopic subsystem represents the system dynamics in a part of flight envelope, and its vertices are the subsystems of a locally overlapped switched system (LOSS) which describes the dynamics on operating points within this part of flight envelope. For every polytopic subsystem, a gain-scheduled subcontroller is achieved by interpolating between the state-feedback controllers on vertices, and gain-scheduled controller with respect to full flight envelope is composed of these gain-scheduled subcontrollers. It is proved that the switched polytopic system is input-to-state uniformly bounded, provided vertices of any polytopic subsystem, i.e., the subsystems of the corresponding LOSS, share a common Lyapunov function, and average dwell time on polytopic subsystems is no smaller than a fixed positive constant.

It is worth noting that system parameters can vary arbitrarily fast in every polytopic subsystem by the proposed approach, due to the existence of common Lyapunov functions for polytopic subsystems. In addition, different from LPV gain-scheduled synthesis methods, the proposed synthesis approach bases on "divide and conquer" philosophy [2] and thus can use large amount of past experience on linear control techniques.

II. Problem Formulation

A. Preliminary Knowledge

A class of switched systems is given by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \tag{1}$$

where $x(t) \in \Re^n$ and $u(t) \in \Re^m$ are the continuous state and the input, respectively, the piecewise constant $\sigma(t)$ is the switching law and takes value discretely from a finite index set Ω at every switching, A_i, $i \in \Omega$ are system matrices and B_i are input matrices.

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For a natural number k, let $\sim^k$ denotes the set $\{1, 2, \cdots, k\}$.

Definition 1: [17] The switching law$\sigma(t)$ is called localizable, if there exist finite index sets Ω_j, $j \in \sim^k$, with the property that $\cup_{s=1}^{k}\Omega_s = \Omega$ and $\forall\ l \in \sim^k$, $\Omega_l \cap (\cup_{s=1,s\neq l}^{k}\Omega_s) \neq \emptyset$, such that $\forall\ t \geq 0$, it holds that $\sigma(t^+) \in \cup_{j\in\{w|\sigma(t)\in\Omega_w\}}\Omega_j$.

In Definition 1, it shows that switched systems with localizable switching law have two interesting properties. One property is that subsystems of the switched system can be partitioned into several locally overlapped groups. The other property is that the switching law evolves in a specific way of either switching within a locally overlapped group or switching to adjacent locally overlapped groups. By these two properties, the switched system (1) can be studied in an equivalent formulation which is specified by the following Definition 2.

Definition 2: [17] If the switching law $\sigma(t)$ evolving upon the switched system is localizable, then $\sigma(t)$ is called as locally overlapped switching law. Further, the switched system (1) can be reformulated as , called localizable formulation, in which $T^j \triangleq \{t|\sigma(t) \in \Omega_j\}$, $j \in \sim^k$ and $\sigma_j(t) \triangleq \sigma(t)$, $t \in T^j$:

$$\begin{cases} \dot{x}(t) = A_{\sigma_1(t)}x(t) + B_{\sigma_1(t)}u(t), & \sigma_1(t) \to \Omega_1 \\ \dot{x}(t) = A_{\sigma_2(t)}x(t) + B_{\sigma_2(t)}u(t), & \sigma_2(t) \to \Omega_2 \\ & \vdots \\ \dot{x}(t) = A_{\sigma_k(t)}x(t) + B_{\sigma_k(t)}u(t), & \sigma_k(t) \to \Omega_k \end{cases} \tag{2}$$

Definitions next are used to detail the structure of the localizable formulation, including the two important definitions of the LOSSs and the common subsystems.

Definition 3: [17] Given the localizable formulation of switched system which evolves on locally overlapped switching law $\sigma(t)$, define

$$\dot{x}(t) = A_{\sigma_j(t)}x(t) + B_{\sigma_j(t)}u(t), \qquad \sigma_j(t) \to \Omega_j, \qquad j \in \sim^k \tag{3}$$

to be LOSS and $\sigma_j(t)$ to be the subswitching law.

Definition 4: [17] Given the localizable formulation of switched system which evolves on locally overlapped switching law $\sigma(t)$, define

$$\dot{x}(t) = A_i x(t) + B_i u(t), \qquad i \in \Omega_{\Im}^{c} = \cap_{\varepsilon\in\Im}\Omega_\varepsilon \tag{4}$$

to be the subset $\Im$ based common subsystem, where $\Im$ is any nonempty subset of $\sim^k$ with the property that $\Omega_{\Im}^{c} \neq \emptyset$.

B. Problem Statement

In this study, the flight dynamics on finite operating points will be described by a switched system with locally overlapped switching law, i.e.,

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \qquad u(t) = K_{\sigma(t)}(r(t) - x(t)) \qquad \sigma(t) \to \Omega \tag{5}$$

where each subsystem is stabilized by a state feedback controller $u(t) = K_i(r(t) - x(t))$ and $r(t) \in \Re^m$ is the command input. The corresponding localizable formulation is

$$\begin{cases} \dot{x}(t) = A_{\sigma_1(t)}x(t) + B_{\sigma_1(t)}u(t), & u(t) = K_{\sigma_1(t)}(r(t) - x(t)), & \sigma_1(t) \to \Omega_1 \\ \dot{x}(t) = A_{\sigma_2(t)}x(t) + B_{\sigma_2(t)}u(t), & u(t) = K_{\sigma_2(t)}(r(t) - x(t)), & \sigma_2(t) \to \Omega_2 \\ & & \vdots \\ \dot{x}(t) = A_{\sigma_k(t)}x(t) + B_{\sigma_k(t)}u(t), & u(t) = K_{\sigma_k(t)}(r(t) - x(t)), & \sigma_k(t) \to \Omega_k \end{cases} \tag{6}$$

and its LOSSs are given by

$$\dot{x}(t) = A_{\sigma_j(t)}x(t) + B_{\sigma_j(t)}u(t), \qquad u(t) = K_{\sigma_j(t)}(r(t) - x(t)) \qquad \sigma_j(t) \to \Omega_j, \qquad j \in \sim^k \tag{7}$$

Every LOSS corresponds to a polytopic system whose vertices are subsystems of the LOSS, and flight dynamics within a part of flight envelope is approximated by this polytopic system. Further, the full envelope flight dynamics is described by a switched polytopic system whose subsystems describe the flight dynamics within different parts of flight envelope, i.e.,

$$\begin{aligned} &\dot{x}(t) = F_{\sigma'(t)}(t)x(t) + G_{\sigma'(t)}(t)u(t) \\ &u(t) = K'_{\sigma'(t)}(t)(r(t) - x(t)), \qquad \sigma'(t) \to \sim^k \\ &F_j(t) = \left\{\sum_{i\in\Omega_j}\alpha_i(t)A_i \middle| \alpha_i(t) \geq 0, \sum_{i\in\Omega_j}\alpha_i(t) = 1\right\} \\ &G_j(t) = \left\{\sum_{i\in\Omega_j}\alpha_i(t)B_i \middle| \alpha_i(t) \geq 0, \sum_{i\in\Omega_j}\alpha_i(t) = 1\right\}, \qquad j \in \sim^k \end{aligned} \tag{8}$$

where K'_j, $j \in \sim^k$ is the gain-scheduled gains and will be synthesized by the state feedback gains K_i, $i \in \Omega_j$.

The objective of this study is to synthesize the gain-scheduled controller

$$u(t) = K'_j(t)(r(t) - x(t)), \qquad j \in \sim^k \tag{9}$$

which guarantees flight stability in each part of flight envelope under arbitrarily fast-varying parameters. Further, analyze full envelope flight stability, associated with uniformly input-to-state boundedness of the switched polytopic system (8), by the common Lyapunov function method and the average dwell time method.

III. Gain-Scheduled Controller Synthesis

In the sequel, a gain-scheduled controller will be synthesized, and full envelope flight stability will be analyzed, which is associated with the stability of the switched polytopic system.

Definition 5: [26] Let $N_\sigma(\Delta T)$ denote the number of discontinuities of a switching law $\sigma(t)$ on an interval ΔT. For given $N_0 \geq 0$, $\tau_a > 0$, $S[\tau_a, N_0]$ stands for the class of switching laws satisfying

$$N_\sigma(\Delta T) \leq N_0 + \frac{\Delta T}{\tau_a}, \qquad \forall\ \Delta T > 0 \tag{10}$$

where τ_a is called the average dwell time.

Lemma 1: [15] The following two statements are equivalent:

1) The polytopic system

$$\dot{x}(t) = F(t)x(t) \tag{11}$$

is robustly asymptotically stable, where $F(t)=\{\sum_{i=1}^{N}\alpha_i(t)A_i|\alpha_i(t) \geq 0, \sum_{i=1}^{N}\alpha_i(t) = 1\}$.

2) The switched system

$$\dot{x}(t) = A_\sigma x(t) \tag{12}$$

is asymptotically stable under arbitrary switching law $\sigma(t) \to \sim^N$.

A. Main Results

Theorem 1: For the switched polytopic system (8), suppose that

1) The vertices of any polytopic subsystem, i.e., the subsystems of the corresponding LOSS, satisfy the linear matrix equalities $(A_i - B_iK_i)^T P_j + P_j(A_i - B_iK_i) = -Q_{ij}$ for every $j \in \sim^k$, $i \in \Omega_j$, where P_j and Q_{ij} are appropriately dimensioned positive symmetric matrices.

2) The gain-scheduled controller of every polytopic subsystem is synthesized as

$$K'_j(t) = G_j^+(t)\left(\sum_{i\in\Omega_j}\alpha_i(t)B_iK_i\right), \qquad j \in \sim^k \tag{13}$$

where $G_j(t)$ is of full row rank, and $G_j^+(t)$ is its Moore-Penrose inverse.

3) The switching law $\sigma'(t) \in S[\tau_a, N_0]$ and the average dwell time τ_a satisfies the inequality

$$\tau_a > \frac{\ell n\, \mu}{\inf\limits_{j\in\sim^k, i\in\Omega_j}[\lambda_{\min}(P_j^{-1}Q_{ij})]}, \qquad \mu = \sup_{j,l\in\sim^k}\left[\frac{\rho_{\max}(P_j)}{\rho_{\min}(P_l)}\right] \tag{14}$$

where $\rho_{\max}(\bullet)$, $\rho_{\min}(\bullet)$ denote the largest singular value and the smallest singular value, respectively, and $\lambda_{\min}(\bullet)$ is the smallest eigenvalue.

Then, the system is uniformly input-to-state bounded under any piecewise bounded command input $r(t)$.

Proof: See the Appendix for more details.

B. Discussions

Remark 1: By Theorem 1, it shows that the flight stability in a part of flight envelope, with respect to the stability of the corresponding polytopic subsystems, is guaranteed by the condition 1 and 2. Condition 1 is a requirement of the existence of a common Lyapunov function for vertices of any polytopic subsystem (i.e., for all the subsystems of the corresponding LOSS). Condition 2 presents a gain-scheduled controller interpolation method. Note that in any part of flight envelope, there is no restriction on variation rate of system parameters and they are permitted to vary arbitrarily fast. Further, the flight stability within the full envelope, associated with the uniformly input-to-state boundedness of switched polytopic system, is ensured by the condition 3 on average dwell time.

Remark 2: In Theorem 1, the requirement of $G_j(t)$ being of full row rank is easy to be satisfied in flight control practice. For longitudinal motion, the stability and maneuverability of the flight vehicles primarily depend on the short period motion. This motion can be described by a second-order state-space model with the two states, angle of attack and pitch rate. In this situation, $G_j(t)$ is of full row rank provided that the controls are no less than the states. Most advanced flight vehicles meet this requirement [27], as they have many control effectors such as elevator, canard and thrust-vectoring engine. For lateral motion, $G_j(t)$ is also of full row rank, for the three states, sideslip angle, yaw rate and roll rate are used to describe the flight dynamics, while at least three effectors, aileron, rudder and thrust-vectoring engine, are implemented to control lateral motion of advanced flight vehicles.

Remark 3: It is worth noting that Theorem 1 still holds for flight vehicles with fewer controls than the states (i.e., $G_j(t)$ is of full column rank), because it can be proved that the system is uniformly input-to-state bounded under interpolation technique as Eq. (13). The deduction of this conclusion bases on the fact that the control effectiveness on the angular variables is rather low [27]. See the Appendix for more details.

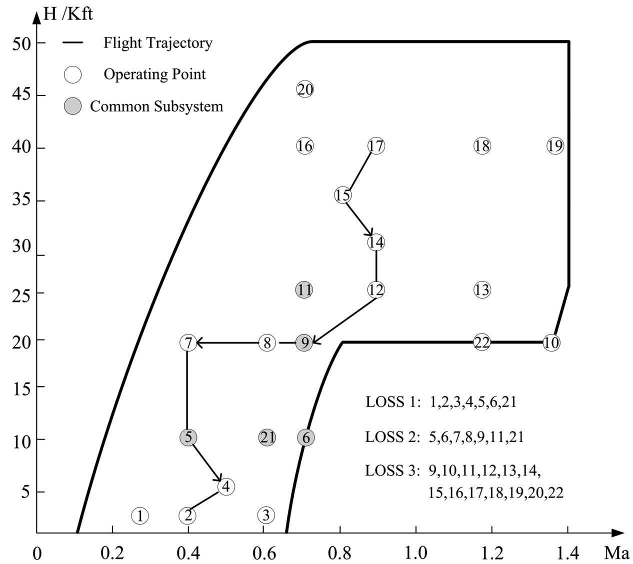

Fig. 1 Flight envelope and operating points of HiMAT vehicle.

IV. Example

The proposed synthesis approach is validated by an application to gain-scheduled control of a highly maneuverable technology (HiMAT) vehicle. Operating points within the full flight envelope is depicted in Fig. 1, and the readers can read [17,28] for detailed descriptions of HiMAT vehicle.

A. Step 1: Switched Polytopic System Description

Within the full flight envelope, flight dynamics of HiMAT vehicle on operating points, indexed by 1 to 22, is described by a localizable switched system. The localizable formulation of the switched system is

$$\begin{cases} \dot{x}(t) = A_{\sigma_1(t)}x(t) + B_{\sigma_1(t)}u(t), & \sigma_1(t) \to \Omega_1 = \{1, 2, 3, 4, 5, 6, 21\} \\ \dot{x}(t) = A_{\sigma_2(t)}x(t) + B_{\sigma_2(t)}u(t), & \sigma_2(t) \to \Omega_2 = \{5, 6, 7, 8, 9, 11, 21\} \\ \dot{x}(t) = A_{\sigma_3(t)}x(t) + B_{\sigma_3(t)}u(t), & \sigma_3(t) \to \Omega_3 = \{9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22\} \end{cases} \tag{15}$$

where $x = (\alpha, q)^T$, and α, q denote the angle of attack and the pitch rate, respectively. $u = (\xi_e, \xi_v, \xi_c)^T$, where ξ_e, ξ_v, ξ_c denote the elevator, the elevon and the canard deflection, respectively. $\sigma(t) \to \Omega = \{1, 2, \cdots, 22\}$ evolves upon Mach and altitude. The designed eigenvalues for operating points $i \in \{1, 2, \cdots, 20\}$ are the same to those in [17], while the eigenvalues on operating points 21 and 22 equal $-4.5 \pm 4.5i$ and $-3.92 \pm 4i$, respectively. The operating points 21–22 are set by the authors to accomplish the interpolation in the following Step 2:

$$A_{21} = 2A_5/3 + A_6/3, \qquad A_{22} = A_{10} \qquad B_{21} = 2B_5/3 + B_6/3$$

$$B_{22} = B_{10}, \qquad K_{21} = \begin{bmatrix} -1.564 & -0.1578 \\ -1.0706 & -0.108 \\ 0.0448 & 0.0045 \end{bmatrix}$$

$$K_{22} = \begin{bmatrix} 2.1523 & -0.0468 \\ 0.7395 & -0.0161 \\ -0.0526 & 0.0011 \end{bmatrix} \tag{16}$$

Based on the localizable formulation (15), the switched polytopic system is constructed as

$$\dot{x}(t) = F_{\sigma'(t)}(t)x(t) + G_{\sigma'(t)}(t)u(t)$$
$$u(t) = K'_{\sigma'(t)}(t)(r(t) - x(t)), \qquad \sigma'(t) \to \{1, 2, 3\} \quad (17)$$

The switched polytopic system (17) is uniformly input-to-state bounded if the gain-scheduled subcontroller are interpolated by Eq. (13) and the average dwell time τ_a on the polytopic subsystems is no smaller than 6.9378 s. This result is achieved by Theorem 1, in which P_j, $j \in \{1, 2, 3\}$ are solved via LMIs $(A_i - B_iK_i)^T P_j + P_j(A_i - B_iK_i) < 0$, and $Q_{ij} = -(A_i - B_iK_i)^T P_j - P_j(A_i - B_iK_i)$. The positive matrices P_j are listed as follows:

$$P_1 = \begin{bmatrix} 0.5439 & -0.0263 \\ -0.0263 & 0.032 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0.662 & -0.0254 \\ -0.0254 & 0.033 \end{bmatrix}$$
$$P_3 = \begin{bmatrix} 0.5112 & -0.0113 \\ -0.0113 & 0.0323 \end{bmatrix} \quad (18)$$

B. Step 2: Gain-Scheduled Controller Synthesis

For every polytopic subsystem, the gain-scheduled subcontroller is interpolated by Eq. (13), e.g., for the region $H \in [2.5, 10)$, $Ma \in [0.4, 0.6)$, the gain-scheduled subcontroller is interpolated by the four gains K_2, K_3, K_5, K_{21} on the operating points 2, 3, 5 and 21. Suppose that a flight condition in this flight region with the altitude h_a and Mach number M_a, then the interpolated gain K' is

$$K' = (\alpha_1 B_2 + \alpha_2 B_3 + \alpha_3 B_5 + \alpha_4 B_{21})^+ (\alpha_1 B_2 K_2 + \alpha_2 B_3 K_3 + \alpha_3 B_5 K_5 + \alpha_4 B_{21} K_{21})$$
$$\alpha_1 = (1 - \lambda_H)(1 - \lambda_{M_a}), \qquad \alpha_2 = (1 - \lambda_H)\lambda_{M_a}$$
$$\alpha_3 = \lambda_H(1 - \lambda_{M_a}), \qquad \alpha_4 = \lambda_H \lambda_{M_a}$$
$$\lambda_H = (h_a - 2.5)/(10 - 2.5), \quad \lambda_{M_a} = (M_a - 0.4)/(0.6 - 0.4) \quad (19)$$

The gain-scheduled controller within full flight envelope consists of all gain-scheduled subcontrollers. Note that switching between gain-scheduled subcontrollers will not induce nonsmooth change of controller gains, provided the controller gains on the boundary of two adjacent polytopic subsystems are only interpolated by the gains on the common subsystems. For example, the controller gains on the boundary $Ma = 0.7$ are only interpolated by the gains on operating points 9 and 11. Similar technique is also adopted in [11].

Remark 4: In this example, the gain-scheduled subcontrollers are interpolated on square regions. The readers can also carry out the interpolation on triangular regions, e.g., by the function “tri2grid” in MATLAB R2007a.

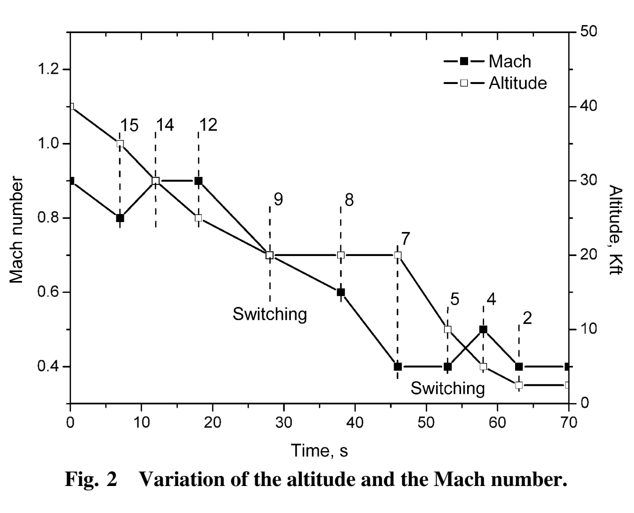

Fig. 2 Variation of the altitude and the Mach number.

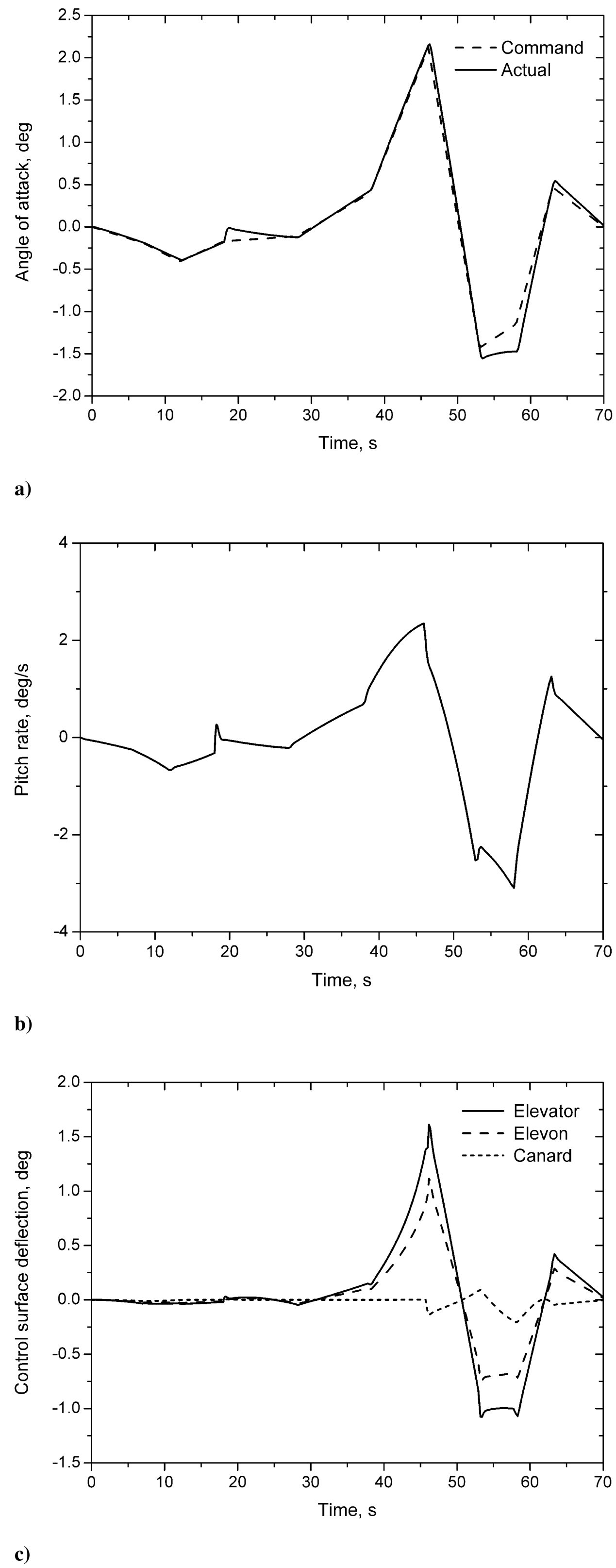

Fig. 3 Trajectories of system states and control inputs: a) trajectory of angle of attack, b) trajectory of pitch rate, and c) control surface deflection.

C. Step 3: Validation

The performance of the gain-scheduled controller is validated by a large-scale flight trajectory, i.e., 17-15-14-12-9-8-7-5-4-2, which is also depicted in Fig. A1. The variation of altitude and Mach number are depicted in Fig. 2. This trajectory includes two switchings between the three polytopic subsystems, one switching occurs when

HiMAT vehicle leaves operating point 9 and the other switching occurs when HiMAT vehicle leaves operating point 5. It can be verified that $\sigma'(t) \in S[6.9378 \text{ s}, N_0]$ for any $N_0 \geq 0$. The pitch rate command is set to be zero during the simulation, while the angle of attack command is depicted in Fig. 3a. The simulation results are depicted in Figs. 3a–3c. By the figures, it can be concluded that the angle of attack tracking performance is acceptable along the entire flight trajectory, and the pitch rate response and control surface deflection are all satisfying over the time history.

V. Conclusions

A switched polytopic system is established to describe flight dynamics within full flight envelope and a gain-scheduled control synthesis approach is proposed based on this flight dynamics description. The switched polytopic system is input-to-state uniformly bounded, provided that the vertices of any polytopic subsystem share a common Lyapunov function, and the average dwell time on polytopic subsystems is no smaller than a fixed positive constant. By the proposed synthesis approach, the system parameters are allowed to vary arbitrarily fast in every polytopic subsystem. Besides, this approach bases on the "divide and conquer" philosophy and thus can use large amount of past design experience on linear control techniques.

Appendix

I. Proof of Theorem 1

Proof: By condition 1, it can be known that each autonomous LOSS

$$\dot{x}(t) = (A_{\sigma_j(t)} - B_{\sigma_j(t)} K_{\sigma_j(t)}) x(t), \qquad \sigma_j(t) \to \Omega_j, \qquad j \in \sim^k \tag{A1}$$

shares a common Lyapunov function

$$V_j(t) = x^T(t) P_j x(t) \tag{A2}$$

and thus each autonomous LOSS is globally asymptotically stable. Further, by Lemma 1, the system

$$\dot{x}(t) = \left(\sum_{i \in \Omega_j} \alpha_i(t) A_i - \sum_{i \in \Omega_j} \alpha_i(t) B_i K_i \right) x(t), \qquad j \in \sim^k \tag{A3}$$

is robustly asymptotically stable.

By Eq. (13) of the condition 2 and Eq. (A3), it can be achieved that the autonomous polytopic subsystem (APS)

$$\dot{x}(t) = (F_j(t) - G_j(t) K'_j(t)) x(t), \qquad j \in \sim^k \tag{A4}$$

is robustly asymptotically stable.

In the sequel, it will be proved that the switchings between APSs will not induce instability.

The common Lyapunov function $V_j(t) = x^T(t) P_j x(t)$ shared by subsystems of the autonomous LOSS is also valid for the jth APS, $j \in \sim^k$ [13].

Let $t_1, t_2, \cdots, t_{N_{\sigma'}(\Delta T)}$ denote the switching times when APSs are consecutively activated in the interval $\Delta T \triangleq t_{N_{\sigma'}(\Delta T)}$, and

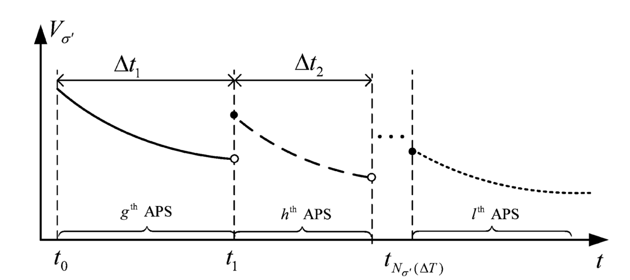

Fig. A1 Switching between APSs with the constraint of average dwell time.

$\Delta t_1, \Delta t_2, \cdots, \Delta t_{N_{\sigma'}(\Delta T)}$ are the dwell time of each APS being active, which is also depicted in Fig. A1.

Because $\mu = \sup_{j,l \in \sim^k} [\rho_{\max}(P_j)/\rho_{\min}(P_l)]$, for two consecutively activated APSs indexed by g and h, it holds that

$$V_h(t_1^+) < \mu V_g(t_1^-) < \mu V_g(t_0) e^{-\eta_g \Delta t_1} \tag{A5}$$

where $\eta_g \triangleq \inf_{t \geq 0} (-\dot{V}_g(t)/V_g(t))$ is decay rate of the Lyapunov function $V_g(t)$, and $\eta_g = \inf_{i \in \Omega_g} [\lambda_{\min}(P_g^{-1} Q_{ig})]$ [29].

Suppose that lth LOSS is activated at $t_{N_{\sigma'}(\Delta T)}$, iterating the technique like Eq. (A5) from 1 to $N_{\sigma'}(\Delta T)$, then

$$\begin{aligned}
V_l(t^+_{N_{\sigma'}(\Delta T)}) &< \mu^{N_{\sigma'}(\Delta T)} V_g(t_0) e^{-\eta_g \Delta t_1 - \eta_h \Delta t_2 - \cdots - \eta_l \Delta t_{N_{\sigma'}(\Delta T)}} \\
&< \mu^{N_{\sigma'}(\Delta T)} V_g(t_0) e^{-\inf_{j \in \sim^k, i \in \Omega_j} [\lambda_{\min}(P_j^{-1} Q_{ij})](\Delta t_1 + \Delta t_2 + \cdots + \Delta t_{N_{\sigma'}(\Delta T)})} \\
&< \mu^{N_{\sigma'}(\Delta T)} V_g(t_0) e^{-\inf_{j \in \sim^k, i \in \Omega_j} [\lambda_{\min}(P_j^{-1} Q_{ij})] \Delta T} \\
&< V_g(t_0) e^{(N_0 + \Delta T/\tau_a) \ln \mu} e^{-\inf_{j \in \sim^k, i \in \Omega_j} [\lambda_{\min}(P_j^{-1} Q_{ij})] \Delta T} \\
&< V_g(t_0) e^{N_0 \ln \mu} e^{(\ln \mu/\tau_a - \inf_{j \in \sim^k, i \in \Omega_j} [\lambda_{\min}(P_j^{-1} Q_{ij})]) \Delta T}
\end{aligned} \tag{A6}$$

By condition 3 on the average dwell time, it can be achieved that

$$\ln \mu/\tau_a - \inf_{j \in \sim^k, i \in \Omega_j} [\lambda_{\min}(P_j^{-1} Q_{ij})] < 0 \tag{A7}$$

Substituting (A7) into (A6), it can be seen that the value of Lyapunov function converges to zero exponentially as $\Delta T \to +\infty$. Namely, the autonomous switched polytopic system

$$\dot{x}(t) = (F_{\sigma'(t)}(t) - G_{\sigma'(t)}(t) K'_{\sigma'(t)}(t)) x(t), \qquad \sigma'(t) \to \sim^k \tag{A8}$$

is globally uniformly asymptotically stable. It should be noted that the uniform property is in the sense over any $\sigma'(t) \in S[\tau_a, N_0]$ but not over any initial time [13].

Because $r(t)$ is piecewise bounded and $G_j(t)$, $K'_j(t)$ are both bounded, the switched polytopic system

$$\begin{aligned}
&\dot{x}(t) = (F_{\sigma'(t)}(t) - G_{\sigma'(t)}(t) K'_{\sigma'(t)}(t)) x(t) + G_{\sigma'(t)}(t) K'_{\sigma'(t)}(t) r(t), \\
&\sigma'(t) \to \sim^k
\end{aligned} \tag{A9}$$

is uniformly input-to-state bounded [14], namely, system (8) is uniformly input-to-state bounded. □

II. Detailed Discussion on Remark 3

For longitudinal motion of the flight vehicles, the control effectiveness on the angle of attack dynamics is rather low, whereas the control effectiveness on the pitch rate is comparatively higher [27]. Hence, the autonomous LOSSj, which describes flight dynamics on the operation points within a part of flight envelope, can be described as

$$\begin{pmatrix} \dot{\alpha} \\ \dot{q} \end{pmatrix} = \left(A_{\sigma_j(t)} - \begin{pmatrix} 0 \\ M^{\delta_e}_{\sigma_j(t)} \end{pmatrix} K_{\sigma_j(t)} \right) \begin{pmatrix} \alpha \\ q \end{pmatrix}$$
$$\sigma_j(t) \to \Omega_j, \qquad j \in \sim^k \tag{A10}$$

where $M_i^{\delta_e}$, $i \in \Omega_j$ is the corresponding dimensional stability derivate. In view that autonomous LOSS j is globally asymptotically stable, by Lemma 1, it can be achieved that the following system is robustly asymptotically stable:

$$\begin{pmatrix} \dot{\alpha} \\ \dot{q} \end{pmatrix} = \left(\sum_{i \in \Omega_j} \alpha_i(t) A_i - \sum_{i \in \Omega_j} \alpha_i(t) \begin{pmatrix} 0 \\ M_i^{\delta_e} \end{pmatrix} K_i \right) \begin{pmatrix} \alpha \\ q \end{pmatrix} \tag{A11}$$

Then,

$$\begin{pmatrix}\dot{\alpha}\\ \dot{q}\end{pmatrix}=\left(\sum_{i\in\Omega_j}\alpha_i(t)A_i-\sum_{i\in\Omega_j}\alpha_i(t)\begin{pmatrix}0\\ M_i^{\delta_e}\end{pmatrix}K_i\right)\begin{pmatrix}\alpha\\ q\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}\dot{\alpha}\\ \dot{q}\end{pmatrix}=\left(\sum_{i\in\Omega_j}\alpha_i(t)A_i-\begin{pmatrix}0&0\\0&1\end{pmatrix}\sum_{i\in\Omega_j}\alpha_i(t)\begin{pmatrix}0\\ M_i^{\delta_e}\end{pmatrix}K_i\right)\begin{pmatrix}\alpha\\ q\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}\dot{\alpha}\\ \dot{q}\end{pmatrix}=\left(\sum_{i\in\Omega_j}\alpha_i(t)A_i-\begin{pmatrix}0\\ \sum_{i\in\Omega_j}\alpha_i(t)M_i^{\delta_e}\end{pmatrix}\right.$$
$$\left.\times\begin{pmatrix}0 & \left(\sum_{i\in\Omega_j}\alpha_i(t)M_i^{\delta_e}\right)^{-1}\end{pmatrix}\left(\sum_{i\in\Omega_j}\alpha_i(t)\begin{pmatrix}0\\ M_i^{\delta_e}\end{pmatrix}K_i\right)\right)\begin{pmatrix}\alpha\\ q\end{pmatrix}\quad\text{(A12)}$$

Further, by Eq. (13), it holds that

$$K_j'(t)=\left(\sum_{i\in\Omega_j}\alpha_i(t)\begin{pmatrix}0\\ M_i^{\delta_e}\end{pmatrix}\right)^{+}\left(\sum_{i\in\Omega_j}\alpha_i(t)\begin{pmatrix}0\\ M_i^{\delta_e}\end{pmatrix}K_i\right)$$
$$=\begin{pmatrix}0 & \left(\sum_{i\in\Omega_j}\alpha_i(t)M_i^{\delta_e}\right)^{-1}\end{pmatrix}\left(\sum_{i\in\Omega_j}\alpha_i(t)\begin{pmatrix}0\\ M_i^{\delta_e}\end{pmatrix}K_i\right)\quad\text{(A13)}$$

Substituting (A13) into (A12), it can be seen that the APS

$$\begin{pmatrix}\dot{\alpha}\\ \dot{q}\end{pmatrix}=\left(\sum_{i\in\Omega_j}\alpha_i(t)A_i-\left(\sum_{i\in\Omega_j}\alpha_i(t)\begin{pmatrix}0\\ M_i^{\delta_e}\end{pmatrix}\right)K_j'(t)\right)\begin{pmatrix}\alpha\\ q\end{pmatrix}\quad\text{(A14)}$$

is robustly asymptotically stable. By the same management in the proof of Theorem 1, it can be proved that switched polytopic system (8) is uniformly input-to-state bounded.

For lateral motion of the flight vehicles, the same conclusion can be achieved by adopting the preceding techniques.

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